MATHEMATICS OF OPERATIONS RESEARCH
Vol. 30, No. 2, May 2005, pp. 420–440
ISSN 0364-765X | EISSN 1526-5471 | 05 | 3002 | 0420
DOI 10.1287/moor.1040.0127

Asymptotic Behavior of Internet Congestion Controllers in a Many-Flows Regime

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Congestion controllers for the Internet are typically designed based on deterministic delay differential equation models. In this paper, we consider the case of a single link accessed by many TCP-like congestion-controlled flows and uncontrolled flows that are modeled as stochastic disturbances. We show that if the number of flows is large and the link capacity is scaled in proportion to the number of users, then under appropriate conditions, the trajectory of the stochastic system is eventually well approximated by the trajectory of a delay-differential equation. Our analysis also throws light on the choice of various parameters that ensure *global asymptotic stability* of the limiting deterministic system in the presence of feedback delay. Numerical examples with some popular congestion feedback mechanisms validate the parameter choices from the analysis. The results indicate that a system with multiple TCP-like flows is globally stable (and thus, that a deterministic model is reasonable if the number of flows is large) as long as *the product of the throughput and feedback delay per flow is not very small*.

Key words: congestion controller; delay-differential equations; mean-flow behavior
MSC2000 subject classification: Primary: 94C99, 37H10; secondary: 60H10, 34D23
OR/MS subject classification: Primary: Communications; secondary: probability
History: Received January 16, 2003; revised November 3, 2003, and April 8, 2004.

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1. Introduction. The design philosophy of the current Internet is based on the end-to-end paradigm, wherein most of the intelligence resides at the end hosts. The network's task is to simply notify the end systems whenever it detects congestion in the network. Congestion detection is based on the *aggregate flow behavior* at the router, and the end hosts are notified by simply dropping or by marking packets using the explicit congestion notification (ECN) bit Floyd [4]. The end host reacts to this information by decreasing its transmission rate, thus adapting to network congestion. In this manner, end-to-end control is maintained using only minimal network resources.

This end-to-end design philosophy has motivated a lot of work Kelly et al. [10], Kunniyur and Srikant [12], Paganini et al. [15] using a utility function maximization framework, leading to a class of end-to-end rate control mechanisms for Internet congestion control. Based on the choice of the utility function, various types of fairness amongst users can be achieved. Further, it has been shown in Kunniyur and Srikant [12] that the congestion-avoidance phase of TCP flow control can be considered a special case of the above framework for appropriately chosen utility functions. Such deterministic rate-based models have led to a better understanding of TCP and also allow us to improve existing congestion control and congestion feedback mechanisms used in the Internet.

Deterministic rate update models that explicitly account for round-trip delay have been the focus of much study in the recent past. An important question that has been addressed deals with stability of network controllers based on these deterministic rate adaptation mechanisms. In Kelly [8], a stability condition for a single proportionally fair congestion controller with delayed feedback was provided. Since then, this result has been extended to networks in Johari and Tan [6], Massoulie [14], and Vinnicombe [19, 20], and in Paganini et al. [15], similar results were shown for a different class of controllers. We also refer the reader to Hollot et al. [5] and Kunniyur and Srikant [13] for other related analysis of congestion controllers with delay. All of the above work dealt with *local stability* of the linearized controllers in the presence of round-trip delay. More recently, sufficient conditions were derived in Deb and Srikant [3] for *global exponential stability* for the case of a single flow accessing a link.

However, one may ask why we should consider *deterministic* models for Internet congestion control. In realistic systems, there are two sources of randomness. First, there can be flows that do not react to congestion control. For instance, these could be *web mice*, which are short flows that terminate before they can react to congestion control. Such uncontrolled flows can be modeled as stochastic disturbances in the router.

Second, the router may only be allowed to toggle the state of a single bit in the packet header to indicate congestion. Under such a protocol, a possible strategy is to *mark* each packet independently with some probability, which is a function of the level of congestion at the link.

In our model, we assume that the actual link price itself is fed back to the source; i.e., more than one bit is available to signal congestion. This could be accomplished if there is a field in the packet for this purpose. An example of such a scheme is given in Katabi et al. [7]. On the other hand, if only one bit of pricing information is allowed, then congestion information can only be conveyed through probabilistic marking. In that case, there is randomness in the price itself. Thus, in this paper we concern ourselves with randomness generated due to uncontrolled short flows only. We refer the reader to Baccelli et al. [2] and Tinnakornsrisuphap and Makowski [18] for analysis of TCP behavior in the context of probabilistic marking at the router. We note that in Tinnakornsrisuphap and Makowski [18], the authors have derived a deterministic model for the average window size of TCP flows with probabilistic marking at the router, but with no propagation delay. Our result can be interpreted as a complementary analysis of the case where the randomness in the system is due to the uncontrolled flows, and the propagation delay (which imposes certain conditions for the limiting deterministic models to be valid) is taken into account. We comment that if the randomness due to the probabilistic marking is ignored, the exact value of the marking probability can be viewed as the link price. Thus, in this paper we use the term *marking function* to mean the *congestion price* as a function of the aggregate arrival rate into the link. The randomness due to probabilistic marking may have an impact on the variance of individual flows in steady state, but not the average arrival rate at a link. A heuristic analysis of the impact of probabilistic marking on individual source-rate variance can be found in Kelly [9, T. J. Ott, unpublished data at http://web.njit.edu/∼ott/Papers/ECN/ECN.pdf], however, a rigorous analysis of the behavior of individual source rates under probabilistic marking and in the presence of nonnegligible feedback delay is lacking, and is not addressed in this paper either.

In Shakkottai and Srikant [16], the authors justified the use of deterministic delay-differential equation models for studying *proportionally fair* congestion controllers. They showed that, in the many-flows regime, the trajectory of the average rate at the router converges to that of the deterministic model with the noise replaced by its mean.

1.1. Main contribution. In this paper, we consider a system consisting of a single link accessed by a large number of *TCP-like flows*, each with identical feedback delay, but with (possibly) different initial conditions, and also accessed by a large number of uncontrolled flows. By a *TCP-like mechanism*, we refer to the rate control model of the congestion-avoidance phase of TCP proposed in Kunniyur and Srikant [12], which is also closely related to the model in Hollot et al. [5]. We are interested in relating this stochastic model to a deterministic model where the noise process is replaced by its mean.

We show that in the presence of uncontrolled flows modeled as stochastic noise, the deterministic delay-differential equation model with the noise replaced by its mean value is accurate in the following sense: The *average rate* of the flows behaves like a single flow *asymptotically in the number of flows and time*. Thus, unlike in the proportionally fair case studied in Shakkottai and Srikant [16], where convergence was shown for each time (as opposed to asymptotically in time), here, the trajectory of the stochastic system *does not* converge to that of the deterministic system in the many-flows regime.

However, if the number of flows is large enough, the *global stability criterion* for a single flow (with minor modifications) is also a global stability condition, in an appropriate sense, for the stochastic system with multiple flows. Thus, the implication is that *parameter design can be carried out using deterministic analysis based on the single-flow model.*

Further, for some standard marking functions used in literature, we show that TCP-like flows with standard TCP parameters satisfy the stability criterion when the bandwidth-delay product (i.e., the product of the throughput and the feedback delay) per flow is sufficiently large.

1.2. Organization of this paper. We begin with a description of the model in §2. We consider multiple TCP-like flows along with uncontrolled flows in the model. In §3, we state the main results of our paper. In the two subsequent sections, we go on to prove these results. Towards this end, in §4 we study a deterministic system by simply considering the mean of the uncontrolled flow rate through the link. We present conditions on the congestion control gain for the global stability of such a system. In §5, we prove the results stated in §3. In §6, we study the conditions derived in the context of standard TCP parameters and give examples to illustrate the results. We conclude in §7.

2. Model, assumptions, and preliminaries.

2.1. System model. Our model is that of a single bottleneck link being accessed by many TCP-like flows. The delay in the forward and the reverse path is $d/2$ (in seconds) so that the round-trip delay of each flow is d s. Such a model can be applicable in a scenario when multiple users behind an ISP access a server through a common bottleneck link as in Figure 1. The number of flows in the system is N, which is also the scaling parameter. We consider a sequence of such systems indexed by N. In the Nth system, there are N flows accessing the link and the capacity of the link is scaled as Nc packets/s so that capacity per flow is maintained at c packets/s. Further, in the Nth system there are N uncontrolled flows accessing the link. Before we describe the rate-update mechanism for the flows, we first comment on the marking function of the link.

The link has a marking function $p(\lambda, C)$ that denotes the fraction of packets marked when the total arrival rate into the link is λ, and the link capacity is C (where $C = Nc$). The marking function is assumed to satisfy the following conditions.

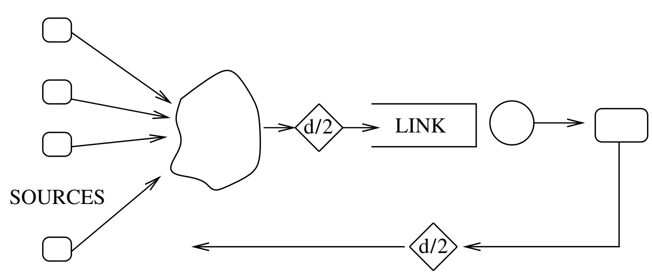

Figure 1. The system model.

Assumption 2.1 (Properties of Marking Function).
(i) *The function* $p(\lambda, C)$ *is increasing in* λ *and is Lipschitz continuous in* λ.
(ii) *We further assume that* $p(\lambda, C) = p(\lambda/C, 1)$.

The first assumption is obvious because $p(\lambda, C)$ is the fraction of packets marked. The second assumption says that the fraction of packets marked simply depends on the ratio of the total arrival rate and the link capacity. To understand this property in the context of our scaling, suppose in the Nth system, that the rate of the ith controlled flow is x_i and the rate of the ith uncontrolled flow is e_i for $1 \leq i \leq N$. Then, the marking function for the Nth system is

$$p(\lambda, Nc) = p\left(\frac{\sum_i (x_i + e_i)}{Nc}, 1\right) = p\left(\frac{x+e}{c}, 1\right),$$

where x and e are the average rate of the controlled and the uncontrolled flows, respectively. Thus, under this assumption, the marking function in the Nth system simply depends on the *average flow rate* through the link for some fixed capacity per flow. Two examples of marking functions that have this property are

$$p(\lambda, C) = \left(\frac{\lambda}{C}\right)^B,$$

$$p(\lambda, C) = \frac{a\lambda}{C - (1-a)\lambda}.$$

The first marking function has the interpretation of the queue size being B or larger in an $M/M/1$ queue with arrival rate x. The second marking function can be used as a rate-based model for a marking scheme called random early marking (REM) Kelly [8]. We remark that any reasonable marking function should satisfy the second assumption. This ensures the scalability of the marking function in the number of flows. Thus, from now on, we will interpret the arguments x and c of the marking function $p(x, c)$ as the average arrival rate and the capacity per flow, respectively. Further, in the systems we consider from now on, the capacity per flow c will remain constant, and the only time-varying parameter is the average rate x. Thus, to avoid unnecessary notation, we will hide the dependence of $p(\cdot, c)$ on c, and let

$$p(x) \equiv p(x, c) = p(x/c, 1).$$

In addition to the controlled flows, we assume that the system is accessed by uncontrolled flows. These are flows that do not react to congestion signals and are modeled as stochastic processes with mean a. In the Nth system, there are N uncontrolled flows $\{e_i^{(N)}[k] + a\}_{i=1}^N$, where $(e_i^{(N)}[k] + a)$ is the rate of the ith uncontrolled flow at kth time epoch in the Nth system. We model $\{e_i^{(N)}[k]\}_{i=1}^N$ as independent and identically distributed (i.i.d.), and bounded stochastic process with mean 0.

We now describe our model for the controlled flows. We consider a fluid model for the rate update of the controlled flows. Denote by $y_i^{(N)}[k]$ the flow rate of the ith flow at time slot k when there are N such flows present in the system. Further, denote by $x^{(N)}[k]$ the average flow rate of the controlled flows through the link at time k, and so

$$x^{(N)}[k] = \frac{1}{N} \sum_{i=1}^{N} y_i^{(N)}[k].$$

Similarly, denote by $(e^{(N)}[k] + a)$ the average flow rate of the N uncontrolled flows through the link. The fraction of packets marked by the link is $p(x^{(N)}[k] + a + e^{(N)}[k])$, where the average flow rate at the link consists of the average flow rate of the controlled flows $x^{(N)}(t)$, and the average flow rate due to N uncontrolled flows $a + e^{(N)}(t)$.

We now describe the time interval over which the flows update their rates. The flows update their rates at discrete time slots. We can view each time slot as a measurement interval over which rates are measured in the system, and control actions by the routers and flows are updated. Typically, this measurement interval is measured in terms of the number of packets that can be processed by a typical router. For example, the time step could be "100 packets long." Thus, by scaling both the time step and the capacity, we can maintain a constant time step, as measured in packets Shakkottai and Srikant [16]. To this end, let each time step in the Nth system be $1/N$. Thus, the update of the ith system at the $(k+1)$th time-step can be described by the following:

$$y_i^{(N)}[k+1]-y_i^{(N)}[k]=\frac{\kappa}{N}\big[w-y_i^{(N)}[k]y_i^{(N)}[k-Nd]p(x^{(N)}[k-Nd]+e^{(N)}[k-Nd]+a)\big],\quad i\in\{1,2,\ldots,N\}. \tag{1}$$

The above equation is a generalized form of the rate-update version of TCP flow control proposed in Kunniyur and Srikant [12]. The right-hand side of the update equation has two parts. If there is no congestion, the packet transmission rate of a flow increases linearly in time. However, if there is congestion in the link, the rate decreases proportional to the rate at which the source receives marked packets, which depends on the rate at which packets were transmitted one round-trip time back. The quantity $y_i^{(N)}[k-Nd]p(x^{(N)}[k-Nd]+e^{(N)}[k-Nd]+a)$ is the rate at which marked packets are received by the source in the kth slot due to the packets sent Nd slots back. Note that because the delay d as measured in seconds is fixed, the delay in the Nth system corresponds to Nd time slots. Further note that we assume that the feedback delay is constant. This is reasonable if we employ early congestion notification schemes using virtual queues that lead to negligible queueing delay at the router.

A continuous-time model can now be embedded as

$$y_i^{(N)}(t)=y_i^{(N)}[Nt]\quad \text{for } Nt\in\mathbb{N},$$

with a straight line approximation used between integers. Similarly, the average rate process $x^{(N)}(t)$ and the noise process $e^{(N)}(t)$ can be defined. Thus, rewriting the rate update as described in (1) using the continuous-time variables, we have the following rate update model in continuous time:

$$\dot{y}_i^{(N)}(t)=\kappa\bigg[w-y_i^{(N)}\bigg(\frac{\lfloor Nt\rfloor}{N}\bigg)y_i^{(N)}\bigg(\frac{\lfloor N(t-d)\rfloor}{N}\bigg)\times p\bigg(x^{(N)}\bigg(\frac{\lfloor N(t-d)\rfloor}{N}\bigg)+a+e^{(N)}\bigg(\frac{\lfloor N(t-d)\rfloor}{N}\bigg)\bigg)\bigg],\quad i\in\{1,2,\ldots,N\}. \tag{2}$$

The above update equation is the same as that described by (1), but written in terms of the continuous-time variables, when we interpret (2) as a representation for the unique trajectory satisfying the integral equation

$$y_i^{(N)}(t)=y_i^{(N)}(0)+\kappa\int_{s=0}^{t}\bigg[w-y_i^{(N)}\bigg(\frac{\lfloor Ns\rfloor}{N}\bigg)y_i^{(N)}\bigg(\frac{\lfloor N(s-d)\rfloor}{N}\bigg)\times p\bigg(x^{(N)}\bigg(\frac{\lfloor N(s-d)\rfloor}{N}\bigg)+a+e^{(N)}\bigg(\frac{\lfloor N(s-d)\rfloor}{N}\bigg)\bigg)\bigg]\,ds.$$

It is in this sense that all the differential equations in the rest of this paper are to be interpreted. We comment that choosing $\kappa=2/3$ and $\kappa w=1/d^2$ in (2) result in the rate control model of standard TCP Kunniyur and Srikant [12].

In the rest of this paper, we find conditions under which the system converges to the unique equilibrium point given by the solution of

$$y_i^* \sqrt{p(y_i^* + a)} = \sqrt{w}$$

in the presence of feedback delay, with and without stochastic disturbance introduced by the uncontrolled flows. We also discuss the implication of our results in the context of TCP.

2.2. Some preliminaries. In this subsection, we discuss two key results that will be useful in some of our derivations.

The first result is on the boundedness of the average trajectory of congestion-controlled flows in the presence of delay and without the stochastic noise. Consider a continuous-time deterministic model of congestion-controlled flows where the noise is modeled by a constant process of rate a. There are N flows and the rate update of the ith flow is given by

$$\dot{y}_i(t) = \kappa[w - y_i(t) y_i(t-d) p(x(t-d) + a)], \quad i = 1, 2, \ldots, N.$$

The trajectory of the average flow rate through the link $x(t)$ can be described by

$$\dot{x}(t) = \kappa \left\{ w - \left[\frac{1}{N} \sum_1^N y_i(t) y_i(t-d) \right] p(x(t-d) + a) \right\}.$$

We are interested in conditions on κ so that the average flow rate $x(t)$ is bounded. The following result, which is derived in Shakkottai et al. [17] provides such a condition.

Lemma 2.1 (Shakkottai et al. [17]). *Suppose that $\kappa d < \beta$. Fix $\delta > 0$, where δ can be arbitrarily small. Then there exists $t_0(\delta) < \infty$ such that for all $t \geq t_0$,*

$$x(t) \leq M_\beta,$$

where M_β is the smallest positive number satisfying

$$M_\beta^2 p(M_\beta + a - 2w)\left(1 - \frac{2w\beta}{M_\beta}\right) \geq w + \delta. \quad \square$$

For the rest of this paper, we will assume that δ in the preceding lemma is a very small number, fixed at, say, $\delta = 0.0001$.

In this paper, we will use a result on global exponential stability of linear time-varying delay differential equations derived in Deb and Srikant [3]. For convenience and completeness of this paper, we state the result below. We will use this in our subsequent analysis of the model.

Lemma 2.2 (Deb and Srikant [3]). *Consider the delay differential equation given by*

$$\dot{x}(t) = a(t)x(t) + b(t)x(t-d)$$

with some initial condition, $x(t) = \phi(t)$, $t \in [-d, 0]$. If there exists $q > 1$ such that $a(t)$ and $b(t)$ satisfy

$$d\sqrt{q} \max_{t-d \leq s \leq t} (|a(s)| + |b(s)|) < -\operatorname{sgn}(b(t)) - \frac{a(t)}{|b(t)|}$$

for all $t \leq t_1$, then

$$V(t) < qV(0)e^{-\alpha t} \quad \forall\, t \leq t_1,$$

where $V(t) = \sup_{t-2d \leq s \leq t} x^2(s)$ and $\alpha > 0$, $q > 1$ are constants. $\square$

3. Main result: Multiple TCP-like flows with identical round-trip time and stochastic noise. Before we state our main result, we state a result on the upper bound of the average flow rate, which is easy to show using the upper bound on the average given in Lemma 2.1. Consider the system in (2). Recall that $x^{(N)}(t)$ denotes the average rate of the flows.

LEMMA 3.1. *Suppose that* $\kappa d < \beta$. *Then, given* $\epsilon' > 0$, *there exist* $\bar{N}$ *and* $\bar{t}(\epsilon')$ *such that* $\forall\,(N \ge \bar{N})$ *and* $\forall\,(t \ge \bar{t})$,

$$x^{(N)}(t) \le M_\beta + \epsilon',$$

where M_β *is as given in Lemma* 2.1. □

We skip the proof of the above lemma. It follows almost immediately from the proof of Lemma 2.1 Shakkottai et al. [17]. Thus, we see that the average rate eventually gets arbitrarily close to M_β for large enough N. Without loss of generality, we can study the system evolution from time $\bar{t}$ onwards. In the following, we thus assume that $x^{(N)}(t) \le M_\beta$ for all $t \in [-d, 0]$ and for all $N \ge 1$. It is more precise to assume that $x^{(N)}(t) \le M_\beta + \epsilon'$ for all $t \in [-d, 0]$ and N large enough. However, our assumption is less notationally cumbersome and does not lead to any loss of generality. Suppose that the initial value of the average rate lies in some compact set $[0, K]$ in which the equilibrium point of the system is included. Then, clearly, there exists a β for which $M_\beta \ge K$. We now assume that the initial condition for each flow satisfies the following.

ASSUMPTION 3.1 (INITIAL CONDITION). *The initial trajectory for any user* $i \in \{1, 2, \ldots, N\}$ *satisfies*

$$M_\beta(1-\epsilon) \le y_i^{(N)}(s) < M_\beta(1+\epsilon) \quad \forall\, s \in [-d, 0]$$

for some $\epsilon < 1$. □

Essentially, this says that the initial values of the individual user rates are not too far away from each other. Because the value of M_β is larger than the equilibrium rate, not allowing the initial user rates to be more than twice the value of M_β is a reasonable assumption. We remark that for all N, the initial conditions are assumed to satisfy the conditions given in Assumption 3.1. We also state below our assumption on the noise process.

ASSUMPTION 3.2 (NOISE PROCESS). *The disturbances due to the uncontrolled flows are bounded; i.e.,* $\exists\, K < \infty$ *such that* $|e_i^{(N)}[k]| < K$ *for all* $i = 1, 2, \ldots, N$. *Further, we assume that these uncontrolled flows are i.i.d., stationary, and ergodic.*

Under Assumption 3.2, it can be shown Shakkottai and Srikant [16] that the average noise process $e^{(N)}(t)$ satisfies

$$\lim_{N\to\infty} \sup_{t\in[-d,\, NT]} |e^{(N)}(t)| = 0 \quad \text{a.s.}$$

Finally, consider the following system given by

$$\dot{u}(t) = \kappa[w - u(t)u(t-d)p(u(t-d)+a)]. \tag{3}$$

This is the rate-update model of single flow accessing a single link with marking function $p(\cdot)$, and, when the noise is modeled by a process of constant rate a. Let R be any value such that the above system (3) is semiglobally exponentially stable[1] for $\kappa d < R$. Techniques for finding such an R are given in Deb and Srikant [3].

[1] A dynamic system with state $x(t)$ and unique equilibrium point x^* is semiglobally exponentially stable if, for $x(0) \in K$ for a compact set K, $\|x(t) - x^*\| \le A(K)\|x(0) - x^*\| \exp(-\gamma t)$ for suitable $A(K) > 0$ and $\gamma > 0$.

We have the following main result of this paper.

THEOREM 3.1. *Suppose that*

$$\kappa d < \min\left[\beta, R, \frac{1}{6M_\beta p(M_\beta + a)}\right].$$

Then, under Assumptions 2.1, 3.1, and 3.2, given $\epsilon' > 0$, $\exists N'(\epsilon')$ such that $\forall N \geq N'$,

$$\left|y_k^{(N)}(NT) - y^*\right| \leq \epsilon' \quad a.s. \qquad k \in \{1, 2, \ldots, N\},$$

where y^* is the solution of $w = y^2 p(y + a)$. □

In the next few sections, we prove the above result. We first establish some intermediate results before we prove Theorem 3.1. One of our results in the next section also provides conditions for a deterministic version of the congestion control model to be globally asymptotically stable.

4. Multiple TCP-like flows and constant rate uncontrolled flows. In this section, we study a deterministic, continuous-time model of congestion control when the flow rate due to the uncontrolled flows is simply modeled by a constant process of rate a. For the purposes of this section, we keep N, the number of flows, fixed. We denote the rate of the ith flow by $y_i(t)$ and the average flow rate by $x(t)$.

The rate update of the ith flow is

$$\dot{y}_i(t) = \kappa[w - y_i(t)y_i(t-d)p(x(t-d)+a)], \quad i = 1, 2, \ldots, N, \tag{4}$$

and the trajectory of the average flow rate through the link $x(t)$ can be described by

$$\dot{x}(t) = \kappa\left\{w - \left[\frac{1}{N}\sum_{1}^{N} y_i(t)y_i(t-d)\right]p(x(t-d)+a)\right\}.$$

Note from Lemma 2.1 that $x(t) \leq M_\beta$ for all $t \geq t_0$. Thus, by shifting the time axis appropriately, we assume that $x(t) \leq M_\beta$ for all $t \geq 0$. Further, the initial trajectory of the individual flows satisfy the conditions given by Assumption 3.1.

Our goal is to find suitable conditions on κ for the system given by (4) to be globally asymptotically stable.

We now introduce the following notation for every pair of flows:

$$r_{ij}(t) = y_i(t) - y_j(t), \quad (i, j) \in \{1, 2, \ldots, N\}^2. \tag{5}$$

Our goal is to show that $r_{ij}(t)$ converges to zero for appropriately chosen κ. This will enable us to show that the system indeed converges to the unique equilibrium point under suitable conditions.

First, note that the dynamics of $r_{ij}(t)$ can be described by the following:

$$\begin{aligned}
\dot{r}_{ij}(t) &= -\kappa p(x(t-d)+a)[y_i(t)y_i(t-d) - y_j(t)y_j(t-d)] \\
&= -\kappa p(x(t-d)+a)[y_i(t)y_i(t-d) - y_i(t)y_j(t-d) + y_i(t)y_j(t-d) - y_j(t)y_j(t-d)] \\
&= -\kappa p(x(t-d)+a)[y_i(t)r_{ij}(t-d) + y_j(t-d)r_{ij}(t)].
\end{aligned}$$

We are now in a position to state and prove the following result on the convergence of $r_{ij}(t)$.

THEOREM 4.1. *If κd satisfies*

$$\kappa d < \min\left[\beta, \frac{1}{6M_\beta p(M_\beta + a)}\right],$$

then

$$\lim_{t\to\infty} \sup_{(i,j)\in\{1,2,\ldots,N\}^2} r_{ij}^2(t) \le A\exp(-\alpha t),$$

where $A > 0$ and $\alpha > 0$ are suitable constants.

REMARK 4.1. Before we go into the details of the proof, we will illustrate the key idea in the proof informally. First, we can show that as long as all the flow rates $\{y_i(t)\}$ are less than $3M$, the difference between their rates $\{r_{ij}(t)\}$ will decrease. This will follow from the single-flow global stability condition from Lemma 2.2. Second, recall from Lemma 2.1 that the average rate is upper bounded as well; i.e., $x(t) < M$ (see Figure 2).

Suppose that at some time, say t_1, it happens that for a particular flow l, $y_l(t_1) = 3M$, and up to time t_1, we have all the individual flow rates strictly less than $3M$. As the average rate at this time $x(t_1) < M$, *there will be some flow k whose rate $y_k(t_1) < M$.*

On the other hand, by assumption, the initial value of $\{r_{ij}(\cdot)\}$ is less than $2M$. From the decreasing property of $\{r_{ij}(\cdot)\}$, it follows that $r_{lk}(t_1) < 2M$. This, along with the fact that $y_k(t_1) < M$ implies that $y_l(t_1) < 3M$, leading to a contradiction. Therefore, we must have that all the flow rates are strictly less than $3M$ for all time, and the required result will follow.

We now formally prove this result.

PROOF OF THEOREM 4.1. Because β is assumed to be fixed throughout the proof, we drop the subscript in M_β, and simply use M in this proof. Define

$$t_1 = \inf_{t>0}\left\{t: \max_{i\in\{1,2,\ldots,N\}}[y_i(t)] \ge 3M\right\}, \tag{6}$$

where we include the possibility of t_1 being infinity (which would mean $\max_i[y_i(t)] < 3M$ for all $t > 0$). Further, define the function $V_{ij}(t)$ as

$$V_{ij}(t) = \sup_{t-2d\le s\le t} r_{ij}^2(s).$$

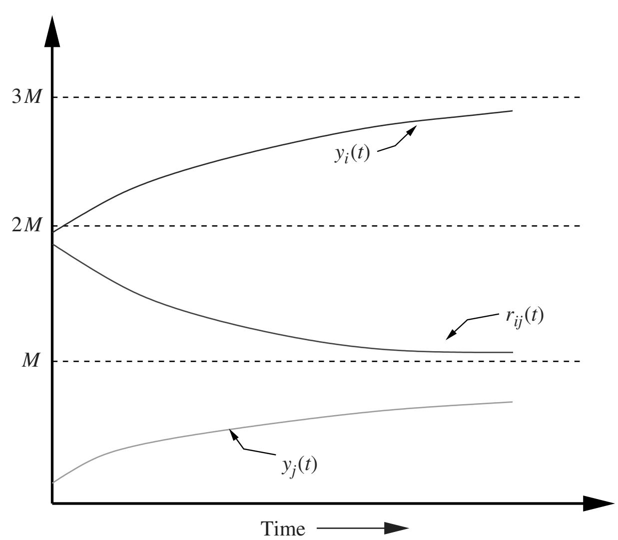

FIGURE 2. Main proof idea in Theorem 4.1.

We divide the proof into two steps. In the first step, we show that under the condition on κd given by the statement of the theorem, $V_{ij}(t) < qV_{ij}(0)\exp(-\alpha t)$ for all $t < t_1$, for every (i, j), and for some constants q and α. In the second step, we show that $t_1 = \infty$ and use this to conclude that $r_{ij}(t)$ converges to zero.

Step 1. Consider the delay-differential equation

$$\dot{r}_{ij}(t) = -\kappa p(x(t-d)+a)[y_i(t)r_{ij}(t-d) + y_j(t-d)r_{ij}(t)]$$

for every pair (i, j). Suppose that

$$\kappa d \max_{t-2d\le s\le t}[p(x(t-d)+a)(y_i(t)+y_j(t-d))] < 1 + \frac{y_j(t-d)}{y_i(t)}.$$

Because we are considering t such that $t < t_1$, we have $0 < y_i(t) < 3M$ for all $i \in \{1, 2, \ldots, N\}$. Further, the average rate $x(t) < M$ by our assumption. Thus, a sufficient condition for the preceding inequality to be satisfied is

$$\kappa d[6Mp(M+a)] < 1.$$

Because we also have $\kappa d < \beta$ for M to be an eventual upper bound on $x(t)$, if κd satisfies the condition given by the statement of the theorem and if q is chosen to satisfy

$$1 < \sqrt{q} < \frac{1}{6Mp(M+a)\kappa d},$$

then by Lemma 2.2,

$$V_{ij}(t) < qV_{ij}(0)\exp(-\alpha t) \quad \forall\, t < t_1,$$

where α is a function of q. In this case, choose q such that

$$1 < \sqrt{q} < \min\left(\frac{1}{\epsilon}, \frac{1}{6Mp(M+a)\kappa d}\right),$$

where ϵ is such that the initial rates of the individual flows lie in $[M(1-\epsilon), M(1+\epsilon)]$. We will consider q in the above range for the rest of the proof.

Step 2. We now show that t_1 as defined by (6) is not finite. We will show it by contradiction. Suppose that $t_1 < \infty$. Because the trajectories of $y_i(t)$s are continuous in t, we have $\max_i[y_i(t_1)] = 3M$. Suppose that

$$k = \underset{i\in\{1,2,\ldots,N\}}{\arg\max}\; y_i(t_1).$$

If κd satisfies the conditions given in the statement of the theorem, we further have from Step 1 that $V_{kj}(t) < qV_{kj}(0)\exp(-\alpha t)$ for all $t < t_1$. Because $V_{kj}(0) < (2M\epsilon)^2$ for all j from our assumption on the initial condition, we have $V_{kj}(t) < q(2M\epsilon)^2\exp(-\alpha t)$ for all $t < t_1$. From the continuity of $V_{kj}(t)$ in t,

$$V_{kj}(t_1) < q(2M\epsilon)^2,$$

which in turn implies

$$\sup_{t_1-2d\le s\le t_1} |r_{kj}(t_1)| < 2M\epsilon\sqrt{q},$$

which further implies

$$|y_k(t_1) - y_j(t_1)| < 2M\epsilon\sqrt{q} \quad \forall\, j.$$

We also have

$$\frac{1}{N}\sum_{i=1}^{N} y_i(t_1) < M$$

because $x(t) < M$ for all $t > 0$. If the average of N quantities is less than M, there must be at least one of them less than M. Let that element be indexed by l so that $y_l(t_1) < M$. Note that $l \neq k$ because $y_k(t_1) = 3M$. We thus have

$$y_k(t_1) \leq |y_k(t_1) - y_l(t_1)| + y_l(t_1) < 2M\sqrt{q}\epsilon + M < 3M, \tag{7}$$

where we have used the fact that $\sqrt{q}\epsilon < 1$. However, $y_k(t_1) = 3M$. Thus, we have arrived at a contradiction, and so for all $t > 0$, $y_i(t) < 3M$ for all $i \in \{1, 2, \ldots, N\}$. This, along with Step 1, implies that

$$V_{ij}(t) < qV_{ij}(0)\exp(-\alpha t) \quad \forall\, t > 0.$$

This proves the convergence of $r_{ij}(t)$. To show that the convergence is uniform in all pairs (i, j), we simply note that the exponent in the exponential convergence only depends on the choice of κ and not on any specific flow. □

Thus, we have shown that the trajectories of all the flows behave almost identically if their round-trip delays are the same. Using this, we can now study the stability with multiple flows by using stability results from the single-flow case. First, note that the average flow rate through the link $x(t)$ can be written as

$$\begin{aligned} x(t) &= \frac{1}{N}\sum_{i=1}^{N} y_i(t) \\ &= y_k(t) + \frac{1}{N}\sum_{i=1}^{N}[y_i(t) - y_k(t)] \\ &= y_k(t) + \delta(t), \end{aligned}$$

where $y_k(t)$ is any particular flow and $\delta(t)$ is a term which goes to zero exponentially. Next, we rewrite the update equation for flow k as follows:

$$\begin{aligned} \dot{y}_k(t) &= \kappa[w - y_k(t)y_k(t-d)p(y_k(t-d)+a+\delta(t-d))] \\ &= \kappa[w - y_k(t)y_k(t-d)p(y_k(t-d)+a)] \\ &\quad + \kappa[y_k(t)y_k(t-d)p(y_k(t-d)+a) - y_k(t)y_k(t-d)p(y_k(t-d)+a+\delta(t-d))] \\ &= \kappa[w - y_k(t)y_k(t-d)p(y_k(t-d)+a)] - y_k(t)y_k(t-d)p'(\beta_k(t)+a)\delta(t-d) \\ \Rightarrow \quad \dot{y}_k(t) &= \kappa[w - y_k(t)y_k(t-d)p(y_k(t-d)+a)] + \eta_k(t). \end{aligned} \tag{8}$$

The second-to-last step follows from the mean-value theorem and $\beta_k(t) = y_k(t-d) + f\delta(t-d)$ for some f such that $0 < f < 1$. Note that because $|\delta(t)| \to 0$ exponentially and all the other terms are bounded, $|\eta_k(t)| \to 0$ exponentially. Thus, we can view the trajectory of the kth flow as a single flow accessing the link except for an additional term, which is negligible for large t. It is thus natural to believe that the stability criterion for

$$\dot{u}(t) = \kappa[w - u(t)u(t-d)p(u(t-d)+a)] \tag{9}$$

is sufficient to guarantee the stability of the system with multiple flows. We show this in our next theorem, which is a simple extension of the global stability result with single flow in Deb and Srikant [3].

THEOREM 4.2. *Suppose that*

$$\kappa d < \min\left[\beta, R, \frac{1}{6M_\beta p(M_\beta + a)}\right],$$

where $R > 0$ is such that $\kappa d < R$ is a sufficient condition for (9) to be globally stable. Then, the system described by (8) is globally exponentially stable.

Before we prove the above result, we state a useful result on the functional differential equation Kolmanovskii and Nosov [11].

LEMMA 4.1 (KOLMANOVSKII AND NOSOV [11, P. 79]). *Consider the retarded functional differential equation*

$$\dot{x}(t) = f(x_t), \quad x_0 = \phi, \tag{10}$$

where $x_t = \{x(t+\theta)\colon -d \le \theta \le 0\} \in CB[-d, 0]$ *and* $\phi \in CB[-d, 0]$. *Assume that* $f\colon CB[-d, 0] \to R^n$ *is continuous, Lipschitz, and* $f(0) = 0$. *Then,* (10) *is exponentially stable if and only if there exists a functional* $V(t, \phi)$ *such that*

$$c_1\|\phi\| \le V(t, \phi) \le c_2\|\phi\|$$
$$\dot{V} \le -c_3\|x_t\| \le -\frac{c_3}{c_2}V, \quad |V(t, \phi) - V(t, \xi)| \le c_4\|\phi - \xi\|,$$

where the norms of the functions are defined as $\|\phi\| = \sup_{-d\le\theta\le 0}|\phi(\theta)|$, *and* c_i *are some positive constants.*

PROOF OF THEOREM 4.2. Because

$$\dot{y}_k(t) = \kappa[w - y_k(t)y_k(t-d)p(y_k(t-d)+a)]$$

is globally exponentially stable Deb and Srikant [3], we have from Lemma 4.1 the existence of a Lyapunov function $V_k(t, y_{k_t})$ such that

$$\dot{V}_k \le -\gamma V_k,$$

and satisfying the properties given in Lemma 4.1. Now, apply the Lyapunov functional

$$V(t) \equiv V(t, y_t) = \frac{1}{N}\sum_{k=1}^{N} V_k(t, y_{k_t}) \tag{11}$$

to the system given by (8). Because

$$\sup_k |\eta_k(t)| \le K_1 \exp(-\alpha t),$$

it is easy to see that

$$\dot{V} \le -\gamma V + K_1 c_4 \exp(-\alpha t),$$

from which it follows that

$$V(t) \le V(0)\exp(-\gamma t) - \frac{K_1 c_4}{\gamma - \alpha}[\exp(-\gamma t) - \exp(-\alpha t)].$$

The result thus follows because all the initial conditions are assumed to lie in a compact set. We note that the exponent in the exponential stability can be chosen as $\min(\gamma, \alpha)$. □

We now use the results derived in this section to prove the main results of this paper, as stated in §3.

5. Proof of the main result in §3. In this section, we show that the global stability criterion of the system given by (4) is sufficient to ensure the stability of the system with stochastic noise in an appropriate sense.

Before we prove our main result, we will prove Theorem 3.1 under the following slightly relaxed assumption:

$$\lim_{N\to\infty} \sup_{t\in[-d,\infty)} |e^{(N)}(t)| = 0 \quad \text{a.s.} \tag{12}$$

We introduce some notation next. Define

$$r_{ij}^{(N)}(t) = y_i^{(N)}(t) - y_j^{(N)}(t), \quad (i,j) \in \{1, 2, \dots, N\}^2. \tag{13}$$

Further, let

$$y_{it}^{(N)} = \{y_i^{(N)}(s)\colon t-d \le s \le t\},$$

and similarly, define $y_t^{(N)}$ for the vector $y^{(N)}(t) = [y_i^{(N)}(t)]_{i=1}^N$. Let

$$\|y_{it}^{(N)}\| = \sup_{t-d\le s\le t} y_i^{(N)}(s).$$

Now note that the update of the ith flow can be written as follows:

$$\begin{aligned}\dot{y}_i^{(N)}(t) &= \kappa\bigg[w - y_i^{(N)}\bigg(\frac{\lfloor Nt\rfloor}{N}\bigg) y_i^{(N)}\bigg(\frac{\lfloor N(t-d)\rfloor}{N}\bigg)\\ &\qquad \times p\bigg(x^{(N)}\bigg(\frac{\lfloor N(t-d)\rfloor}{N}\bigg) + a + e^{(N)}\bigg(\frac{\lfloor N(t-d)\rfloor}{N}\bigg)\bigg)\bigg]\\ &= \kappa\bigg[w - y_i^{(N)}(t) y_i^{(N)}(t-d)\\ &\qquad \times p\bigg(x^{(N)}\bigg(\frac{\lfloor N(t-d)\rfloor}{N}\bigg) + a + e^{(N)}\bigg(\frac{\lfloor N(t-d)\rfloor}{N}\bigg)\bigg)\bigg] + g_i\big(y_t^{(N)}, y_{\lfloor Nt\rfloor/N}^{(N)}\big),\end{aligned}$$

where

$$\begin{aligned}g_i\big(y_t^{(N)}, y_{\lfloor Nt\rfloor/N}^{(N)}\big) &= \kappa y_i^{(N)}(t) y_i^{(N)}(t-d) p\bigg(x^{(N)}\bigg(\frac{\lfloor N(t-d)\rfloor}{N}\bigg) + a + e^{(N)}\bigg(\frac{\lfloor N(t-d)\rfloor}{N}\bigg)\bigg)\\ &\quad - \kappa y_i^{(N)}\bigg(\frac{\lfloor Nt\rfloor}{N}\bigg) y_i^{(N)}\bigg(\frac{\lfloor N(t-d)\rfloor}{N}\bigg)\\ &\quad \times p\bigg(x^{(N)}\bigg(\frac{\lfloor N(t-d)\rfloor}{N}\bigg) + a + e^{(N)}\bigg(\frac{\lfloor N(t-d)\rfloor}{N}\bigg)\bigg).\end{aligned}$$

Now we note that the argument of $p(\cdot)$ is bounded. Further, because $(t - \lfloor Nt\rfloor/N) \le 1/N$, by applying the mean-value theorem to the expression for $g_i(y_t^{(N)}, y_{\lfloor Nt\rfloor/N}^{(N)})$, it can be shown after some straightforward algebraic manipulations that

$$\big|g_i\big(y_t^{(N)}, y_{\lfloor Nt\rfloor/N}^{(N)}\big)\big| \le \frac{1}{N} h_i\big(\|y_{it}^{(N)}\|, \|y_{i\lfloor Nt\rfloor/N}^{(N)}\|, \|y_{i\lfloor N(t-d)\rfloor/N}^{(N)}\|\big),$$

where $h_i(.,.,.)$ is a polynomial in $\|y_{it}^{(N)}\|$, $\|y_{i\lfloor Nt\rfloor/N}^{(N)}\|$, and $\|y_{i\lfloor N(t-d)\rfloor/N}^{(N)}\|$.

As before, it can be shown for (2) that

$$\begin{aligned}\dot{r}_{ij}^{(N)}(t) &= -\kappa p\bigg(x^{(N)}\bigg(\frac{\lfloor N(t-d)\rfloor}{N}\bigg) + a + e^{(N)}\bigg(\frac{\lfloor N(t-d)\rfloor}{N}\bigg)\bigg)\\ &\qquad \times \Big[y_i^{(N)}(t) r_{ij}^{(N)}(t-d) + y_j^{(N)}(t-d) r_{ij}^{(N)}(t)\Big]\\ &\qquad + g_i\big(y_t^{(N)}, y_{\lfloor Nt\rfloor/N}^{(N)}\big) - g_j\big(y_t^{(N)}, y_{\lfloor Nt\rfloor/N}^{(N)}\big).\end{aligned} \tag{14}$$

We also remind the reader that for all N, the initial conditions are assumed to satisfy the conditions given in Assumption 3.1. We have the following result.

THEOREM 5.1. *Suppose that the noise process satisfies* (12). *If*

$$\kappa d < \min\bigg[\beta, \frac{1}{6M_\beta p(M_\beta + a)}\bigg],$$

then, given $\epsilon' > 0$, $\exists\,(\bar{t}(\epsilon'), \bar{N})$ *such that* $\forall\,(N \geq \bar{N})$,

$$\sup_{t\in[\bar{t},\infty)} \big(r_{ij}^{(N)}(t)\big)^2 < \epsilon' \quad a.s..$$

Proof. Without loss of generality, we assume that $\epsilon' < M_\beta \epsilon$. (Recall that the initial condition is such that $|r_{ij}^{(N)}(0)| < 2M_\beta\epsilon$.) The proof of this result is very much similar to the proof of Theorem 4.1. First, we outline the key differences from Step 1 of the proof of Theorem 4.1.

As in Step 1 of the proof of Theorem 4.1, first let us consider $t \leq t_1$, where

$$t_1 = \inf_{t>0}\{t : \max_{i\in\{1,2,\ldots,N\}} [y_i^{(N)}(t)] \geq 3M\}.$$

Clearly, there exists $\delta > 0$ such that

$$\kappa d < \min\left[\beta, \frac{1}{6(M_\beta+\delta)p(M_\beta+\delta+a)}\right]. \tag{15}$$

Because $x^{(N)}(s) \leq M_\beta$ for $s \in [-d, 0]$, we have from Lemma 3.1 that $\exists\, N_1$ such that $\forall\, N \geq N_1$,

$$\sup_{t\in[0,\infty)} x^{(N)}(t) \leq M_\beta + \frac{\delta}{2}.$$

Further, $\exists\, N_2$ such that $\forall\, N \geq N_2$,

$$\sup_{t\in[0,\infty)} |e^{(N)}(t)| < \delta/2.$$

Let $\bar{N} = \max\{N_1, N_2\}$. Note that for $t \in [0, \infty)$, the following is true:

$$\kappa d < \frac{1}{6\left(M_\beta + \frac{\delta}{2}\right)p\left(M_\beta + \frac{\delta}{2} + a + e^{(N)}\left(\frac{\lfloor N(t-d)\rfloor}{N}\right)\right)} \tag{16}$$

whenever (15) holds. We first look at the following slightly different system given by

$$\dot{\hat{r}}_{ij}^{(N)}(t) = -\kappa p\left(x^{(N)}\left(\frac{\lfloor N(t-d)\rfloor}{N}\right) + a + e^{(N)}\left(\frac{\lfloor N(t-d)\rfloor}{N}\right)\right) \times \big[y_i^{(N)}(t)\hat{r}_{ij}^{(N)}(t-d) + y_j^{(N)}(t-d)\hat{r}_{ij}^{(N)}(t)\big],$$

for which it can be seen from Theorem 4.1 that $\sup_{t-d\leq s\leq t} |\hat{r}_{ij}(t)|^2 < q|\hat{r}_{ij}(0)|^2$ for $t < t_1$ provided (16) holds. This can be proved using Razumikhin's theorem by showing that, under (16), if

$$q\big(\hat{r}_{ij}^{(N)}(t)\big)^2 > \max_{s\in[t-2d,t]} \big(\hat{r}_{ij}^{(N)}(t)\big)^2,$$

then

$$\frac{d}{dt}\big(\hat{r}_{ij}^{(N)}(t)\big)^2 \leq -K_1\big(\hat{r}_{ij}^{(N)}(t)\big)^2,$$

where $q > 1$, $K_1 > 0$.

Redoing a similar calculation for the system given by (14) and for $t < t_1$, we can show that whenever

$$q\big(r_{ij}^{(N)}(t)\big)^2 > \max_{s\in[t-2d,t]} \big(r_{ij}^{(N)}(t)\big)^2,$$

then

$$\frac{d}{dt}\big(r_{ij}^{(N)}(t)\big)^2 \leq -K_1\big(r_{ij}^{(N)}(t)\big)^2 + \frac{12M_\beta}{N}|h_i(3M_\beta, 3M_\beta, 3M_\beta) + h_j(3M_\beta, 3M_\beta, 3M_\beta)|.$$

The factor $12M_\beta$ accounts for the fact that for $t < t_1$, $|r_{ij}^{(N)}(t)| < 6M_\beta$. Choose N large enough so that $12M_\beta|h_i(3M_\beta, 3M_\beta, 3M_\beta) + h_j(3M_\beta, 3M_\beta, 3M_\beta)|/N < K_1\epsilon'^2/2$. It follows that

$$\frac{d}{dt}\big(r_{ij}^{(N)}(t)\big)^2 < -\frac{K_1\epsilon'^2}{2}$$

whenever

$$q\big(r_{ij}^{(N)}(t)\big)^2 > \max_{s\in[t-2d,\,t]}\big(r_{ij}^{(N)}(t)\big)^2, \quad \big(r_{ij}^{(N)}(t)\big)^2 > \epsilon'^2, \qquad t < t_1.$$

Using an argument similar to Step 2 of Theorem 4.1, we can show that t_1 is unbounded. The result follows from Razumikhin's theorem for ultimate boundedness. □

Note that from the preceding result we immediately have that $y_i^{(N)}(t) < M_\beta + \epsilon'$ for all $t > \bar{t}$ and N large enough. By doing calculations similar to that in (8), the update for the ith flow can be written as

$$\dot{y}_i^{(N)}(t) = \kappa\big[w - y_i^{(N)}(t)y_i^{(N)}(t-d)p\big(y_i^{(N)}(t-d)+a\big)\big] + \delta_{kN}(t), \quad k\in\{1,2,\ldots,N\}, \tag{17}$$

where, given $\epsilon' > 0$, $\exists\,\bar{N}(\epsilon')$, and $\bar{t}(\epsilon')$ such that

$$\sup_{t\in[\bar{t},\,\infty)} |\delta_{kN}(t)| < \epsilon' \quad \forall\, N \geq \bar{N}.$$

We now have a result similar to Theorem 3.1.

Lemma 5.1. *Suppose*

$$\kappa d < \min\left[\beta, R, \frac{1}{6M_\beta p(M_\beta + a)}\right].$$

If the noise process satisfies (12), *then under Assumptions* 2.1 *and* 3.1, *given* $\epsilon' > 0$, $\exists\,(\bar{t}(\epsilon'), \bar{N})$ *such that* $\forall\,(N \geq N')$,

$$\sup_{t\in[\bar{t},\,\infty)} |y_k^{(N)}(t) - y^*| \leq \epsilon' \quad a.s. \quad k \in \{1, 2, \ldots N\},$$

where y^* *is the solution of* $w = y^2p(y + a)$.

Proof. As in the proof of Theorem 4.2, consider the Lyapunov functional given by (11) for the system given by (17). It can be seen that

$$\dot{V} \leq -\gamma V + c_4\delta_N(t),$$

where $\lim_{N\to\infty}\lim_{t\to\infty}|\delta_N(t)| = 0$. The result is easy to show from this by considering all times $t > \bar{t}(\epsilon')$ and sufficiently large N. □

We now prove the main result of the paper, which follows from the preceding lemma along with minor additional arguments.

Proof of Theorem 3.1. Fix a typical sample path $e^{(N)}(t)$, and also fix ϵ'. Consider a process $e'^{(N)}(t)$ satisfying

$$\sup_{t\in[0,\,\infty)} e'^{(N)}(t) \to 0$$

and

$$e'^{(N)}(t) = e^{(N)}(t) \quad \forall\,(t \leq NT).$$

Note that the update of the ith flow can be written as

$$\begin{aligned}\dot{y}_i^{(N)}(t) = \kappa\Bigg[w - y_i^{(N)}\bigg(\frac{\lfloor Nt\rfloor}{N}\bigg)y_i^{(N)}\bigg(\frac{\lfloor N(t-d)\rfloor}{N}\bigg)\\ \times p\bigg(x^{(N)}\bigg(\frac{\lfloor N(t-d)\rfloor}{N}\bigg) + a + e'^{(N)}\bigg(\frac{\lfloor N(t-d)\rfloor}{N}\bigg)\bigg)\Bigg]\\ + K(t)\bigg|e'^{(N)}\bigg(\frac{\lfloor N(t-d)\rfloor}{N}\bigg) - e^{(N)}\bigg(\frac{\lfloor N(t-d)\rfloor}{N}\bigg)\bigg|,\end{aligned}$$

where $K(t)$ is a bounded process. The statement of the theorem thus follows from Lemma 5.1 because the $y_i^{(N)}(t)$ process behaves as if the noise process is $e'^{(N)}(t)$ for $t \le NT$. □

6. Stability conditions with standard TCP. In previous sections, we have seen that the stability condition with a single TCP connection accessing a link along with some additional condition is enough to ensure the stability and the convergence of multiple TCP-like flows in a many-flows regime.

In this section, we use the earlier results to derive stability conditions with the standard TCP parameters. Because the global stability criterion of a single TCP flow accessing a bottleneck link plays an important role in the many-flows regime, we first state the stability condition with a single TCP flow with parameters implied by standard TCP. Recall that the congestion avoidance phase of TCP can be modeled as in Kunniyur and Srikant [12],

$$\dot{x}(t) = \frac{1}{d^2} - \frac{2}{3} x(t) x(t-d) p(x(t-d)+a, c), \tag{18}$$

where x is in segments (each segment may correspond to 512 bytes) per time unit and a is the mean flow rate due to the uncontrolled flows. We define $W(t)$ as

$$W(t) = d x(t).$$

The evolution of the congestion-avoidance phase of TCP can be rewritten as

$$\dot{W}(t) = \frac{2}{3d}\left[\frac{3}{2} - W(t)W(t-d)p\left(\frac{W(t-d)+ad}{cd}, 1\right)\right]. \tag{19}$$

Because $x(t)$ is the rate in segments per unit time and d is the round-trip time, $W(t)$ can be interpreted as the congestion window size in segments. Further, the quantity cd determines the desired bandwidth-delay product per flow at the equilibrium. Thus, (19) is a continuous-time version of the window update algorithm of the TCP algorithm in the congestion-avoidance phase.

The following lemma provides sufficient conditions under which (19) is globally exponentially stable. The condition follows from the results derived in Deb and Srikant [3]. We state the stability condition of (19) below. For the rest of this section, we let

$$q(w) = p\left(\frac{w}{cd}, 1\right).$$

LEMMA 6.1. *The controller given by* (19) *is globally exponentially stable if*

$$\frac{2}{3} < \frac{l^2(3/2 + M^2 R)}{M^3 R(3/2 + l^2 R)},$$

where $R = \sup_{l \le w \le M}(q(w) + wq'(w))$, M *is the smallest positive number satisfying*

$$M(M-1)p(M-1+ad) \ge \frac{3}{2},$$

and l *is the largest positive number satisfying*

$$l\left(l + \frac{2}{3}M^2 q(M+ad) - 1\right) q\left(l + \frac{2}{3}M^2 q(M+ad) - 1 + ad\right) \le \frac{3}{2}. \quad \square$$

Now we consider multiple TCP flows. We further consider only the mean flow rate of the uncontrolled flows, because the stability of such a system is sufficient to ensure the stability of the system with stochastic disturbances when the number of flows is large. Let there be N flows, with the update equation of the ith flow described by

$$\dot{W}_i(t) = \frac{2}{3d}\left[\frac{3}{2} - W_i(t)W_i(t-d)q(W(t-d)+ad)\right], \quad i \in \{1, 2, \ldots N\}, \tag{20}$$

where $W(t)$ is the average window size ($W(t) = dx(t)$ where $x(t)$ is the average flow rate) of the N flows. Using Theorem 4.2, we can easily derive stability conditions for the system described by (20). The following result provides such conditions.

Corollary 6.1. *The system given by* (20) *is globally stable if the following conditions hold*:

(i) *We have* $(2/3)M_1 q(M_1 + a) < 1/6$, *where* M_1 *is the smallest positive number satisfying*

$$(M_1 - 1)^2 q(M_1 + ad - 2)\left(1 - \frac{1}{M_1 - 1}\right) \geq \frac{3}{2}.$$

(ii) *The condition given by Lemma* 6.1 *is satisfied.* □

We next present examples for two different marking functions to demonstrate the usefulness of the previous result. We are interested in finding the range of the equilibrium bandwidth-delay product to ensure stability and convergence of multiple TCP flows. We now demonstrate that, for the examples considered, we can ensure global stability if the bandwidth-delay product is large enough.

Example 6.1 ($M/M/1$ Type Marking Function). We consider the marking function

$$p(x, c) = \left(\frac{x}{c}\right)^B = \left(\frac{w}{cd}\right)^B. \tag{21}$$

Here, x is the average flow rate of the flows through the link, and c is a parameter that can be adjusted for a desired bandwidth per flow at the equilibrium. Such a marking function has the interpretation of probability of the buffer size being larger than B in an $M/M/1$ queue with arrival rate x. The equilibrium rate per flow x^* in this case is given by

$$x^* d = \left(\frac{3}{2}\right)^{1/(B+2)} (cd)^{B/(B+2)}.$$

We are interested in finding values of equilibrium bandwidth-delay product x^*d to guarantee global stability of multiple TCP flows.

We show that the conditions given by Corollary 6.1 are satisfied as $x^*d \to \infty$. To make our calculations easy, we assume that $(cd)^{B/(B+2)} > 5$ for reasons that will become obvious soon. In other words, we seek values of x^*d in the range $[5(3/2)^{1/(B+2)}, \infty)$ to ensure global stability of multiple identical TCP-like flows. For the given marking function, first note that for M given in Lemma 6.1, we have

$$\begin{aligned} M &< 1 + (cd)^{B/(B+2)}\left(\frac{3}{2}\right)^{1/(B+2)} \\ &< (cd)^{B/(B+2)}\left[\frac{1}{5} + \left(\frac{3}{2}\right)^{1/(B+2)}\right] = K(cd)^{B/(B+2)}, \end{aligned}$$

where

$$K = \frac{1}{5} + \left(\frac{3}{2}\right)^{1/(B+2)}.$$

It can also be easily seen that

$$M^2 q(M) < K^{B+2}.$$

Also, for l given in Lemma 6.1, we have

$$\begin{aligned} l &> 1 + (cd)^{B/(B+2)} \left(\frac{3}{2}\right)^{1/(B+2)} - \frac{2}{3} M^2 p(M) \\ &> (cd)^{B/(B+2)} \left[\left(\frac{3}{2}\right)^{1/(B+2)} - \frac{1}{5}\left(\frac{2}{3}K^{B+2} - 1\right) \right]. \end{aligned}$$

It follows from some algebraic manipulations that Condition (ii) in Corollary 6.1 is satisfied when

$$M\left(\frac{3}{2l^2} + R\right) < \frac{3}{2},$$

which is satisfied if

$$(cd)^{B/(B+2)} \geq \frac{2}{3} K^{B+1}(1+B) + \frac{K}{\left[\left(\frac{3}{2}\right)^{1/(B+2)} - \frac{1}{5}\left(1 - \frac{2}{3}K^{B+2}\right) \right]^2}.$$

Next, note that M_1, the upper bound on the average rate in Condition (i) of Corollary 6.1, is such that

$$M_1 < \left(\frac{3}{2}\right)^{1/(B+2)} (cd)^{B/(B+2)} + 2 < K_1 (cd)^{B/(B+2)},$$

where

$$K_1 = \frac{2}{5} + \left(\frac{3}{2}\right)^{1/(B+2)}.$$

One can check that a sufficient condition for Condition 1 in Corollary 6.1 to be satisfied is

$$(cd)^{B/(B+2)} \geq 4K_1^{B+1}.$$

Because $x^* d = (3/2)^{1/(B+2)} (cd)^{b/(B+2)}$, a sufficient global stability criterion for multiple TCP-like flows is

$$x^* d \geq \left(\frac{3}{2}\right)^{1/(B+2)} \max\left[\frac{2}{3} K^{B+1}(1+B) + \frac{K}{\left((3/2)^{1/(B+2)} - (1/5)\left(1-(2/3)K^{B+2}\right)\right)^2}, 4K_1^{B+1}, 5 \right].$$

It follows that the system is globally stable as $x^* d \to \infty$.

With a more accurate numerical calculation based on Corollary 6.1, it can be verified that for $B = 8$, the system is globally stable for

$$x^* d \geq 16.66,$$

which corresponds to at least eight packets per flow at the equilibrium because each packet consists of approximately two segments. For $B = 5$, a sufficient condition is

$$x^* d \geq 13.43,$$

corresponding to at least seven packets per flow at the equilibrium.

One can obtain similar results for the marking function

$$p(x) = \frac{(x/c)^B (1 - (x/c))}{1 - (x/c)^{B+1}},$$

which can viewed as the blocking probability in an $M/M/1/B$ queue with arrival rate x and service rate c. A sufficient condition with $B=8$ is

$$x^*d \geq 14.97,$$

and with $B=5$,

$$x^*d \geq 12.44$$

provides a sufficient condition for global stability.

Example 6.2 (Random Early Marking or REM). Next, we consider the following marking function:

$$p(x, c) = \frac{\theta\sigma^2 x}{\theta\sigma^2 x + 2(c-x)}, \tag{22}$$

where σ^2 denotes the variability of the traffic and c can be tuned to obtain a desired rate allocation at equilibrium. For our purposes, we can simply assume that marking takes place at a rate given by (22). However, according to Kelly [8], under appropriate conditions (22) can be viewed as an approximation to a well-known marking mechanism called REM Athuraliya et al. [1].

In this example, we take $\theta\sigma^2 = 0.5$. The equilibrium rate allocation can be obtained by solving

$$\frac{0.5(x^*d)^3}{0.5x^*d + 2(cd - x^*d)} = \frac{3}{2}.$$

It can be verified that

$$x^*d \leq (6cd)^{1/3}$$

First, we argue that the global stability condition is satisfied as $x^*d \to \infty$. Suppose we consider cd in the range $cd \geq 5$, which corresponds to $x^*d \geq 2.63$. We are interested in finding x^*d in the range $[2.37, \infty)$ to guarantee stability. Note that the parameter M in Lemma 6.1 satisfies

$$K_1(cd)^{1/3} \leq M \leq K_2(cd)^{1/3}$$

for suitable constants K_1 and K_2. Further, it can be shown that

$$l \geq K_3(cd)^{1/3}$$

and

$$R = \sup_{l \leq w \leq M} (q(w) + wq'(w)) \leq K_4(cd)^{-2/3}$$

for appropriate positive constants K_3 and K_4. It follows that Condition (ii) in Corollary 6.1 is satisfied when

$$M\left(\frac{3}{2l^2} + R\right) < \frac{3}{2},$$

which is satisfied if

$$(cd)^{1/3} \geq \frac{2}{3}K_2\left(\frac{1}{K_3} + K_4\right).$$

Similarly, Condition (i) in Corollary 6.1 can be expressed as

$$(cd)^{1/3} \geq K_5.$$

Because $x^*d \leq (6cd)^{1/3}$, a sufficient condition for global stability in this case is

$$x^*d \geq K$$

for a suitable constant K.

One can use numerical calculations to obtain a sufficient condition for global stability with multiple TCP flows as

$$x^* d \geq 8.67.$$

This corresponds to at least five packets per flow at the equilibrium, with packet sizes of two segments.

The above examples clearly indicate the following: *For reasonable marking functions and large enough target bandwidth-delay product per flow, multiple TCP flows eventually behave like a single flow and the system is globally asymptotically stable.*

7. Conclusions. We have studied a system consisting of a single link accessed by a large number of TCP-like flows, each with identical delay access, but with (possibly) different initial condition and also accessed by a large number of uncontrolled flows. The contributions of this paper are:

(i) Our main result is that, in the presence of uncontrolled flows (stochastic noise), if the number of flows is large enough, the global exponential stability criterion for a single flow (with minor modifications) is also a *global stability* condition for the stochastic system with multiple flows. Thus, the implication is that *parameter design can be carried out using deterministic analysis based on the single-flow model.*

(ii) For the rate adaptation model of TCP Kunniyur and Srikant [12], we have shown that the stability is ensured if the target equilibrium delay-bandwidth product (window size) per flow is large enough, and we have derived bounds on this quantity. Thus, we have derived sufficient conditions for global stability. Numerical examples with two popular marking functions indicate that the target window size per flow required to ensure stability is not very large.

Acknowledgments. This research was supported by AFOSR URI F49620-01-1-0365, DARPA Grant F30602-00-2-0542, and was partially supported by NSF Grants ACI-0305644 and CNS-0325788.

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